

# Convergence or generic divergence of Birkhoff normal form

Ricardo Perez-Marco\*

**Abstract.** *We prove that Birkhoff normal form of hamiltonian flows at a non-resonant singular point with given quadratic part are always convergent or generically divergent. The same result is proved for the normalization mapping and any formal first integral.*

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## Introduction

In this article we study analytic (**R** or **C**-analytic) hamiltonian flows

$$\begin{aligned}\dot{x}_k &= +\frac{\partial H}{\partial y_k} \\ \dot{y}_k &= -\frac{\partial H}{\partial x_k}\end{aligned}$$

where  $x_k, y_k \in \mathbf{C}$  (resp.  $\mathbf{R}$ ),  $k = 1, 2, \dots, n$ , and  $H$  is an analytic hamiltonian with power series expansion at 0 beginning with quadratic terms (so 0 is a singular point of the analytic vector field). We shall restrict our attention to those  $H$  having a non-resonant quadratic parts: If  $(\lambda_1, \dots, \lambda_n)$  are the eigenvalues of the symmetric matrix  $Q$  where  $\frac{1}{2}(x, y)Q(x, y)^t$  is the quadratic part of  $H$  then, defining  $\lambda_{n+1} = -\lambda_1, \dots, \lambda_{2n} = -\lambda_n$ , there is no relation of the form

$$i_1 \lambda_1 + \dots + i_{2n} \lambda_{2n} = 0$$

with integral coefficients  $i_1, \dots, i_{2n}$  except for the trivial case  $i_1 = \dots = i_{2n} = 0$ . Due to some confusion that one finds in some of the litterature on the problem of convergence of Birkhoff normal form and Birkhoff transformation, we start with a brief historical overview.

The normal form of a hamiltonian flow near a singular point has been studied since the origins of mechanics. The long time evolution of the system near the equilibrium position is better controlled in variables osculating those of the normal form that corresponds to a completely integrable system. This idea is at the base of many computations in Celestial

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\* UCLA, Dept. of Mathematics, 405, Hilgard Ave., Los Angeles, CA-90095-1555, USA,  
e-mail: ricardo@math.ucla.edu.

Mechanics. Its importance, both practical and theoretical, cannot be underestimated. One can consult the reference memoir "Les méthodes nouvelles de la mécanique céleste" by H. Poincaré ([Po]) to get an idea of the central place that the perturbative approach played already in the XIXth century.

Assuming that the eigenvalues of the quadratic part of  $H$  present no resonances, we have a simple formal normal form. This result goes back to C.E. Delaunay [De] and A. Lindstedt [Li] (see [Po], [Si2]). Nowadays this normal form is named after Birkhoff. Birkhoff normal form is the starting point of most of the studies of stability near the equilibrium point: the first studies by E.T. Whittaker [Wh], T.M. Cherry [Ch], G.D. Birkhoff [Bi1, Bi2], and C.L. Siegel [Si1] [Si2], K.A.M. theory ([Ko], [Ar], [Mo]), Nehoroshev's diffusion estimates [Ne],...

The dream of an analytic conjugacy to the normal form (without fixing the quadratic part of  $H$ ) was quickly dissipated after the work of H. Poincaré ([Po] vol.I chapitre V). Poincaré's divergence theorem is the starting point of his difficult proof of the inexistence of non-trivial local first integrals in the three body problem.

Research then focused in understanding the divergence of the conjugation mapping (normalization mapping) with a fixed non-resonant quadratic part for  $H$ . The normal form is unique. The normalization mapping is not unique, but appropriate normalizations determine it uniquely. Different results showed with increasing strength that the normalization mapping was generically divergent. We refer to the book of C.L. Siegel and J. Moser ([Si-Mo] chapter 30) for an overview. The strongest result on divergence being proved by Siegel in 1954 ([Si2]) showing the generic divergence of the normalization, the quadratic part of the hamiltonian being fixed but otherwise arbitrary. A.D. Bruno [Br] and H. Rüssman [Ru2] [Ru3] proved the convergence of the normalization when Birkhoff normal form for the hamiltonian is quadratic and the eigenvalues satisfy Bruno's arithmetic condition (other proofs can be found in [El2], [E-V], [Sto1], [Sto2]).

Despite this progress, the most natural question remains untouched. The question is not the convergence or divergence of the normalizing map, but actually the convergence or divergence of Birkhoff normal form itself. If in first place Birkhoff normal form is diverging, then there is no point in trying to conjugate to the normal form. Also in this case the normalization is necessarily diverging.

Very surprisingly, there seems to be no significant results on this fundamental question. It appears to be a very hard question. The author first learned about it from H. Eliasson. The references in the literature are scarce. H. Eliasson points out in the introduction of his article [El1] that

*"...if the normal form itself is convergent or divergent is not known..."*,

and he points out in [El2]

*"...Generically (...) the formal transformation is divergent. (if the normal form itself also is generically divergent is not known)."*

These are the only citations in the literature that the author is aware of (despite the title of [It] what is really proved there is the convergence of the normalization). On the

other hand, one frequently finds in some litterature the claim "Birkhoff normal form is generically diverging" in place of the "Birkhoff transformation is generically diverging"...

More surprisingly, not a single example is known of an analytic hamiltonian having a divergent Birkhoff normal form. The main result in this article is that the existence of a single example with divergent Birkhoff normal form forces generic divergence. To be more precise we need to introduce the notion of  $\Gamma$ -capacity of a subset of  $\mathbf{C}^n$ . This notion generalizes the notion of capacity in dimension 1. We recall the definition in section 2. We refer the reader to [Ro]. An important property, as in dimension 1, is that a set  $E \subset \mathbf{C}^n$  with zero  $\Gamma$ -capacity is Lebesgue and Baire thin, i.e.  $E$  has zero Lebesgue measure and is of the first category (a countable union of nowhere dense sets).

In order to talk about generic properties we define a natural Baire space. We consider the Banach space  $\mathcal{H}$  of Hamiltonians with radius of convergence 1 endowed with a uniform norm in some open subset of the disk of convergence. Similar results hold for  $\mathbf{C}$ -analytic and  $\mathbf{R}$ -analytic hamiltonians.

We can now state:

**Theorem 1.** *We consider the subspace of  $\mathcal{H}_Q \subset \mathcal{H}$  of analytic hamiltonians*

$$H = \sum_{l=2}^{+\infty} H_l$$

*with fixed non-resonant quadratic part  $H_2$  given by the symmetric matrix  $Q$ .*

*If there exists one hamiltonian  $H_0 \in \mathcal{H}_Q$  with divergent Birkhoff normal form (resp. normalization), then a generic hamiltonian in  $\mathcal{H}_Q$  has divergent Birkhoff normal form (resp. normalization).*

*More precisely, all hamiltonians in any complex (resp. real) affine finite dimensional subspace  $V$  of  $\mathcal{H}_Q$  have a convergent Birkhoff normal form (or normalization), or only an exceptional subset in  $V$  of  $\Gamma$ -capacity 0 (resp. of Lebesgue measure 0) have this property.*

Observe that the second scenario holds for all affine subspaces containing  $H_0$ . The result obtained in the real analytic case is stronger than stated. When  $V$  is a one real dimensional affine line, the exceptional set has zero capacity in the complexification of  $V$ . So the exceptional set has even Hausdorff dimension zero.

The important issue that remains unsettled is thus the existence of hamiltonians with diverging Birkhoff normal form for any non-resonant quadratic part. It seems to be the prevalent opinion among specialist that there is generic divergence for all non-resonant quadratic parts. This feeling is probably motivated by the divergence results on the normalization, which, it is worth noting, are independent of the quadratic part. The author sees no reason against the convergence of Birkhoff normal forms, in particular when the eigenvalues of the quadratic part of  $H$  enjoy good arithmetic properties. Fixing the quadratic part of the hamiltonian, the answer may depend on the arithmetic of its eigenvalues.

On the other hand, using standard methods of Small Divisors, it is not difficult to exhibit hamiltonians with diverging normalizations using Liouville eigenvalues for the

quadratic part. Combining this construction with the previous theorem, one recovers with a simple proof Siegel's result ([Si2]) on the generic divergence of the normalization mapping for some fixed quadratic parts.

Note that fixing the quadratic part of the hamiltonian makes the problem much harder, not allowing to take any advantage of the arithmetic of the eigenvalues. One can find in the litterature results without fixing the quadratic part ([Po] volume I chapter V, [Koz]). One may ask about the reason for studying hamiltonians with fixed quadratic part. Note that for systems with particles, the masses enter directly into the quadratic part of the hamiltonian through the kinetic energy. Thus if one, for example, wants to show the non-integrability of a given system *with given masses* then families of hamiltonians with fixed quadratic part arise naturally. One can cite at this respect the strict criticism of A. Wintner to Poincaré's proof of non-integrability of the three body problem ([Wi] 320, p.241):

*...Poincaré has established a result which concerns the non-existence of additional integrals (...) Nevertheless, his result, as well as its formal refinement obtained by Painlevé, is not satisfactory (...) In fact, these negative results do not deal with the case of fixed, but rather with unspecified, values of the masses  $m_i$  (...) Clearly, these assumptions in themselves do not allow any dynamical interpretation, since a dynamical system is determined by a fixed set of positive numbers  $m_i$  ...*

Without sharing this extreme view, one cannot deny some point in Wintner's criticism.

We prove a second theorem on the divergence of first integrals. The classical approach to integrability of hamiltonian systems is based on *first integrals*. A first integral  $P$  is a convergent power series in the  $2n$  variables  $x_1, \dots, y_n$  such that

$$\{P, H\} = 0$$

where the Poisson bracket is defined by

$$\{P, H\} = \sum_{k=1}^n \left( \frac{\partial P}{\partial x_k} \frac{\partial H}{\partial y_k} - \frac{\partial P}{\partial y_k} \frac{\partial H}{\partial x_k} \right) .$$

The equation  $\{P, H\} = 0$  is equivalent to  $\dot{P} = 0$ , that is to the conservation of  $P$ . By E. Noether's theorem, symmetries of the hamiltonian generate first integrals. Two first integrals,  $P_1$  and  $P_2$ , are in involution (or functionally independent) if their Poisson bracket vanishes

$$\{P_1, P_2\} = 0 .$$

At a non-singular point of the hamiltonian, Liouville's theorem shows that the hamiltonian system is integrable by quadratures if there exists  $n$  first integrals in involution. The case of a non-resonant singular point as considered here is more involved. It has been shown by H. Rüssman [Ru1] for  $n = 2$  and in general by J. Vey [Ve] and H. Ito [It] that the existence of  $n$  first integrals in involution forces the convergence of the normalization to

Birkhoff normal form (H. Eliasson settled the analogue of Vey's theorem in the  $C^\infty$  case [El1], [El3]). Recently L. Stolovitch found a unified approach to Bruno's theorem cited before and Vey's and Ito's theorems ([Sto1], [Sto2]). Once all symmetries of a system have been used to find first integrals in involution, the natural question is if there are any others. Multiple approaches to non integrability have been developed starting from H. Poincaré. We refer to [Koz] for an overview of classical methods. R. de la Llave has recently found that Poincaré's conditions are necessary and sufficient for uniform integrability ([Ll], see also the paper by G. Gallavotti [Ga]). We refer to [Mo] for an account on recent methods of S.L. Ziglin, J. Morales Ruiz and J.-P. Ramis. In the smooth non-analytic setting we refer to the work of R.C. Robinson ([Rob]).

It is natural to define the *degree of integrability* of a hamiltonian as the maximal number  $1 \leq \iota(H) \leq n$  of first integrals in involution. When the normalization is convergent, we have that  $\iota(H) = n$ , so the study of convergent first integrals can be seen as a refinement of the study of the convergence of the normalization.

**Theorem 2.** *We consider the space  $\mathcal{H}_Q$ . Given a hamiltonian  $H_0 \in \mathcal{H}_Q$ , we have for a generic hamiltonian  $H \in \mathcal{H}_Q$ ,*

$$\iota(H) \leq \iota(H_0) .$$

*More precisely, let  $P$  be a universal formal first integral. In any complex (resp. real) affine finite dimensional subspace  $V$  of  $\mathcal{H}_Q$  all hamiltonians  $H \in V$  have converging  $P(H)$ , or only an exceptional set in  $V$  of  $\Gamma$ -capacity zero (resp. Lebesgue measure zero) have this property.*

We give in section 1 a precise definition of universal formal first integral. This theorem reduces the proof of the generic divergence of a given formal first integral in a family of hamiltonians, to the divergence for one hamiltonian. Also, given a family  $V$ , the minimum degree of integrability in  $V$ ,

$$\iota_V = \min_{H \in V} \iota(H)$$

is attained for a generic  $H \in V$ .

The families  $V$  in theorem 1 and 2 can be more general than finite dimensional affine subspaces. The same proof gives the results for example when  $V$  is parametrized polynomially by  $\mathbf{C}^m$ . It is interesting to note how in these theorems the complexification of the problem sheds new light on the real analytic case.

The main idea of this article has also been applied to other problems of small divisors ([PM1], [PM2]).

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## 1) Birkhoff normal form and first integrals.

### a) Birkhoff normal form.

We review briefly in this section the construction of Birkhoff normal form. We follow [Si-Mo]. We need to pay particular attention on the polynomial dependence of the transformation and Birkhoff normal form on the original coefficients of the hamiltonian function. More precisely, it is important for our purposes to keep track of the degrees of the polynomial dependence. We use the sub-index notation for partial derivatives.

We consider an analytic hamiltonian (**R** or **C** analytic)

$$H(x, y) = \sum_{l=2}^{+\infty} H_l(x, y)$$

where  $H_l$  is the homogeneous part of degree  $l$  in the real or complex variables  $x_1, \dots, x_n, y_1, \dots, y_n$ . We can assume, by means of a preliminary linear change of variables, that  $H_2$  is already in diagonal form ([Bi] section III.7)

$$H_2(x, y) = \sum_{k=1}^n \lambda_k x_k y_k .$$

We look for a simpler normal form of the system

$$\begin{aligned} \dot{x}_k &= H_{y_k} \\ \dot{y}_k &= -H_{x_k} \end{aligned}$$

We consider symplectic transformations that leave unchanged the hamiltonian character of the system of differential equations. The new variables  $(\xi, \eta)$  are related to the old ones  $(x, y)$  by the canonical transformation

$$\begin{aligned} x_k &= \varphi_k(\xi, \eta) = \xi_k + \sum_{l=2}^{+\infty} \varphi_{kl}(\xi, \eta) \\ y_k &= \psi_k(\xi, \eta) = \eta_k + \sum_{l=2}^{+\infty} \psi_{kl}(\xi, \eta) \end{aligned}$$

where the  $\varphi_{kl}$  and  $\psi_{kl}$  are the homogeneous parts of degree  $l$ . These canonical transformations are defined by a generating function

$$v(x, \eta) = \sum_{l=2}^{+\infty} v_l(x, \eta)$$

where  $v_l$  is the homogeneous part of degree  $l$ , and  $v_2(x, \eta) = \sum_{k=1}^{+\infty} x_k \eta_k$ . Then the canonical transformation is defined by the equations

$$\begin{aligned}\xi_k &= v_{\eta_k}(x, \eta) = x_k + \sum_{l=3}^{+\infty} v_{l, \eta_k}(x, \eta) \\ y_k &= v_{x_k}(x, \eta) = \eta_k + \sum_{l=3}^{+\infty} v_{l, x_k}(x, \eta)\end{aligned}$$

So we get

$$\begin{aligned}x_k &= \xi_k - \sum_{l=3}^{+\infty} v_{l, \eta_k}(\varphi(\xi, \eta), \eta) \\ y_k &= \eta_k + \sum_{l=3}^{+\infty} v_{l, x_k}(\varphi(\xi, \eta), \eta)\end{aligned}$$

and

$$\begin{aligned}\varphi_{kl}(\xi, \eta) &= -v_{l+1, \eta_k}(\xi, \eta) - \left\{ \sum_{j=3}^l v_{j, \eta_k}(\varphi(\xi, \eta), \eta) \right\}_l \\ \psi_{kl}(\xi, \eta) &= v_{l+1, x_k}(\xi, \eta) + \left\{ \sum_{j=3}^l v_{j, x_k}(\varphi(\xi, \eta), \eta) \right\}_l\end{aligned}$$

where  $\{.\}_l$  indicates that we take the  $l$  homogeneous part of the expression within brackets. From these expressions we have that the coefficients of  $\varphi_{kl}$  and  $\psi_{kl}$  are polynomials with integer coefficients on the coefficients of  $v_3, \dots, v_l, v_{l+1}$ .

To each coefficient of  $v_l$  we assign a degree  $l - 2$  (as we will see next, we will choose a canonical transformation so that the coefficients of the  $v_l$ 's are polynomials on the coefficients of  $H$  of degree  $l - 2$  at most). By induction, we show that the degree of  $\varphi_{kl}$  is at most  $l - 1$ . For  $l = 2$  it is clear. Then by induction, the degree of the coefficients of the homogeneous part of degree  $l$  of an homogeneous monomial

$$\prod_{k=1}^n (\varphi(\xi, \eta))^{\alpha_k} \eta^{\beta_k}$$

of total degree  $j$  ( $\sum \alpha_k + \sum \beta_k = j$ ) is at most  $l - j$ . Thus the degree of

$$v_{j, \eta_k}(\varphi(\xi, \eta), \eta)$$

is at most  $(j - 2) + (l - j + 1) = l - 1$ , and this finishes the induction. The same discussion applies to  $\psi$  and the coefficient  $\psi_{kl}$  has degree  $l - 1$ .

Now the canonical transformation generated by  $v$  transforms the differential system into

$$\begin{aligned}\dot{\xi}_k &= K_{\eta_k} \\ \dot{\eta}_k &= -K_{\xi_k}\end{aligned}$$

where

$$K(\xi, \eta) = \sum_{l=2}^{+\infty} H_l(\varphi(\xi, \eta), \psi(\xi, \eta)) = \sum_{l=2}^{+\infty} K_l(\xi, \eta)$$

where  $K_l$  is the  $l$ -homogeneous part.

Our aim is to construct a canonical transformation which gives a hamiltonian  $K$  only depending on power series of the products  $\omega_k = \xi_k \eta_k$ . The coefficients of  $v$  are constructed by induction on the degree  $l$  of the homogeneous part. Assume that the choices for  $v_3, \dots, v_{l-1}$  have been done so that the new hamiltonian has monomials of order  $\leq l-1$  only depending on the  $\omega_k$ 's. We consider a monomial of degree  $l$

$$P = \prod_{k=1}^n \xi_k^{\alpha_k} \eta_k^{\beta_k}.$$

We want to choose the coefficient  $\gamma$  of  $P$  in  $v_l(\varphi(\xi, \eta), \eta)$  such that the new hamiltonian does not contain the monomial  $P$ . Note that

$$K_l(\xi, \eta) = \sum_{k=1}^{+\infty} \lambda_k (\xi_k v_{lx_k}(\varphi(\xi, \eta), \eta) - \eta_k v_{l\eta_k}(\varphi(\xi, \eta), \eta)) + A$$

where the first term comes from the expansion of  $H_2(\phi(\xi, \eta), \psi(\xi, \eta))$  and the second term  $A$  collects everything coming from higher order. The coefficients in the expression  $A$  are polynomials in the coefficients of  $v_3, \dots, v_{l-1}$  and linear functions in the coefficients of  $H_3, \dots, H_l$ .

By induction we prove at the same time that the coefficients of  $v_l$  are polynomials of degree  $l-2$  on the coefficients of  $H_3, \dots, H_l$ , and also the coefficients of  $K_l$  are polynomials of degree  $l-2$  on the coefficients of  $H_3, \dots, H_l$ . Assuming the induction hypothesis, we have as before that the right hand side in the above formula for  $K_l$  is a polynomial of degree  $\leq l-2$  on the coefficients of  $H_3, \dots, H_l$ .

Now we have

$$\sum_{k=1}^n \lambda_k (\xi_k P_{\xi_k} - \eta_k P_{\eta_k}) = \left( \sum_{k=1}^n \lambda_k (\alpha_k - \beta_k) \right) P$$

Thus if  $\lambda = \sum_{k=1}^n \lambda_k (\alpha_k - \beta_k) \neq 0$ , choosing

$$\gamma = -\frac{1}{\lambda} \{A\}_P$$

(where brackets indicate that we extract the  $P$  monomial) the new hamiltonian will not contain the monomial  $P$ . Note that by the non-resonance condition,  $\lambda = 0$  only happens when  $\alpha_k = \beta_k$  for  $k = 1, \dots, n$ . In that way we determine all coefficients of  $v_l$  except those of the monomials which are a product of  $\omega_k$ 's. Note also that by induction these coefficients are polynomials on the coefficients of  $H_3, \dots, H_l$  of degree  $\leq l-2$ .



In order to determine the coefficients of  $v_l$  for the remaining monomials one takes the normalization that no product of powers of  $\omega_k$ 's appears in

$$\Phi = \sum_{k=1}^n (\xi_k y_k - \eta_k x_k)$$

when expressed in  $(\xi, \eta)$  variables. One checks that this determines uniquely  $v$  and thus the canonical transformation that transforms the hamiltonian into its Birkhoff normal form. When  $H$  is real analytic, it is easy to check ([Si-Mo]) that the previous construction yields a real formal canonical transformation and a real Birkhoff normal form. We summarize this discussion in the following proposition.

**Proposition 1.1.** *Given a hamiltonion flow*

$$\begin{aligned}\dot{x}_k &= H_{y_k} \\ \dot{y}_k &= -H_{x_k}\end{aligned}$$

*with  $H(x, y) = \sum_{l=2}^{+\infty} H_l(x, y)$  with non-resonant quadratic part  $H_2$ , there exists a unique formal canonical transformation defined by a formal generating series*

$$v(x, \eta) = \sum_{l=2}^{+\infty} v_l(x, \eta)$$

*such that in the new variables  $(\xi_k, \eta_k)$  the differential system takes the form*

$$\begin{aligned}\dot{\xi}_k &= K_{\eta_k} \\ \dot{\eta}_k &= -K_{\xi_k}\end{aligned}$$

*where the new hamiltonian  $K$  is a formal power series in the products  $\omega_k = \xi_k \eta_k$ , and the expression*

$$\Phi = \sum_{k=1}^n (\xi_k y_k - \eta_k x_k) .$$

*contains no product of the  $\omega_k$  in the  $(\xi, \eta)$  variables. Moreover, the coefficients of the homogeneous part of  $K$  of degree  $l$  and of  $v_l$  are polynomials of degree  $l-2$  in the coefficients of  $H_3, \dots, H_l$ .*

## **b) First integrals.**

We review some classical facts about first integrals (see [Si1]).

If the normalization is converging, then all expresions

$$\omega_k = \xi_k \eta_k$$

are first integrals since

$$\{\omega_k, K\} = \eta_k K_{\eta_k} - \xi_k K_{\xi_k} = \xi_k \eta_k (K' - K') = 0 .$$

Expressing  $\omega_k$  in terms of the initial variables  $(x, y)$  we get  $n$  formal first integrals

$$P_k(x, y) = \xi_k(x, y)\eta_k(x, y) .$$

Observe that

$$\eta_k = y_k - \sum_{l=3}^{+\infty} v_{l, x_k}(x, \eta) .$$

So if

$$\eta_k(x, y) = y_k + \sum_{l=2}^{+\infty} \eta_{kl}(x, y)$$

where  $\eta_{kl}$  is the  $l$ -homogeneous part of  $\eta$ , then by induction the coefficients of  $\eta_{kl}$  are polynomial on the coefficients of  $H_3, \dots, H_{l+1}$  of degree  $l - 1$ .

We reach the same conclusion for  $\xi_k$  using

$$\xi_k(x, y) = v_{\eta_k}(x, \eta) = x_k + \sum_{l=3}^{+\infty} v_{l, \eta_k}(x, \eta) .$$

Now, we have the following formal lemma ([Si1] lemma 1):

**Lemma 1.2.** *Any formal integral  $P$  can be represented as a formal power series in the  $n$  first integrals  $\omega_1, \dots, \omega_n$ .*

**Proof.** Let  $P(x, y)$  be a formal first integral. We have that

$$P(x, y) = \hat{P}(\xi, \eta)$$

is a formal first integral in the  $(\xi, \eta)$  variables. We write

$$\hat{P} = T + J$$

where  $T$  is the formal power series containing all monomials of the form

$$\prod_{k=1}^n \xi_k^{\alpha_k} \eta_k^{\alpha_k} ,$$

thus  $T$  is a formal power series on the  $n$  formal first integrals  $\omega_1, \dots, \omega_n$ . We only need to show that  $F$  is identically 0. If not consider the leading monomial of  $J$

$$L = \prod_{k=1}^n \xi_k^{\alpha_k} \eta_k^{\beta_k} ,$$

with some  $\alpha_k - \beta_k \neq 0$ . The formal power series  $J$  is a formal first integral, and computing the leading term in

$$\begin{aligned}
0 = \{J, K\} &= \sum_{k=1}^{+\infty} J_{\xi_k} K_{\eta_k} - J_{\eta_k} K_{\xi_k} \\
&= \sum_{k=1}^{+\infty} (J_{\xi_k} \xi_k - J_{\eta_k} \eta_k) K' \\
&= \sum_{k=1}^{+\infty} ((\alpha_k - \beta_k) L + \dots) K'
\end{aligned}$$

we get

$$\sum_{k=1}^{+\infty} \lambda_k (\alpha_k - \beta_k) = 0 .$$

So by the non-resonance condition, for  $k = 1, \dots, n$ ,  $\alpha_k - \beta_k = 0$  and  $J = 0$ .  $\diamond$

Thus we can identify the set of formal first integrals with the formal power series in  $n$  variables.

**Definition 1.3.** *A universal formal first integral  $P(H)$  is  $P(H) = F(\omega_1, \omega_n)$  where  $F$  is a formal power series in  $n$  variables.*

**Corollary 1.4.** *Any universal formal first integral  $P(H)$  has coefficients that are monomials of degree  $l$  depending polynomially on the coefficients of  $H_3, \dots, H_{l+1}$  with degree  $\leq l - 1$ .*

## 2) Proof of the theorems.

### a) Potential theory.

#### $\Gamma$ -capacity.

We recall the definition of  $\Gamma$ -capacity and we refer to [Ro] for more properties. Let  $E \subset \mathbf{C}^m$ . The  $\Gamma$ -projection of  $E$  on  $\mathbf{C}^{m-1}$  is the set  $\Gamma_m^{m-1}(E)$  of  $z = (z_1, \dots, z_{m-1}) \in \mathbf{C}^{m-1}$  such that

$$E \cap \{(z, w) \in \mathbf{C}^m\}$$

has positive capacity in the complex plane  $\mathbf{C}_z = \{(z, w) \in \mathbf{C}^m\}$ . We define

$$\Gamma_m^1(E) = \Gamma_2^1 \circ \Gamma_3^2 \circ \dots \Gamma_m^{m-1}(E) .$$

Finally, the  $\Gamma$ -capacity is defined as

$$\Gamma\text{-Cap}(E) = \sup_{A \in U(m, \mathbf{C})} \text{Cap } \Gamma_m^1(A(E)) .$$

where  $A$  runs over all unitary transformations of  $\mathbf{C}^m$ .

The following lemma is useful ([Ro] Lemma 2.2.8 p.92)

**Lemma.** *Let  $E \subset \mathbf{C}^m$ ,  $E \neq \mathbf{C}^m$  and assume that the intersection of  $E$  with any complex line which is not a subset of  $E$  has inner capacity zero. Then the  $\Gamma$ -capacity of  $E$  is zero.*

As we will see, the set of elements in  $\mathcal{H}$  with convergent Birkhoff normal form (or normalization) is an  $F_\sigma$ -set, so capacitable, and the inner capacity is the capacity of the set. Thus using this lemma, we are reduced to prove the second assertion of the theorem only when the sub-space  $V$  of  $\mathcal{H}$  has dimension 1.

#### Bernstein lemma.

The following is a classical lemma in potential theory and approximation theory ([Ra] p.156). It plays a crucial role in the proof of theorem 1.

**Lemma (Bernstein).** *Let  $E \subset \mathbf{C}$  be a non-polar compact set (i.e.  $\text{cap}(E) > 0$ ). Let  $\Omega$  be the connected component of  $\overline{\mathbf{C}} - E$  containing  $\infty$ . Then for any polynomial  $P$  of degree  $n$ , we have for  $t \in \mathbf{C}$ ,*

$$|P(t)| \leq e^{ng_\Omega(t, \infty)} \|P\|_{C^0(K)}$$

where  $g_\Omega$  denotes the Green function of  $\Omega$ .

The proof is quite simple, we include it here for completeness.

**Proof.** We can assume the polynomial monic. Then

$$u(t) = \log P(t) - \log \|P\|_{C^0(K)} - g_\Omega(t, \infty)$$

is sub-harmonic, is negative near  $\infty$  (because  $g_\Omega(t, \infty) = \log |t| + \text{cap}(E) + o(1)$ ), and  $\limsup u(t) \leq 0$  when  $t \rightarrow K$ . The application of the maximum principle concludes the proof.  $\diamond$

### b) Proof of theorem 1.

The assertion about the divergence of the normalization mapping follows the same lines than the case of the Birkhoff normal. The convergence or divergence of the normalizing transformation is equivalent to the convergence or divergence of the generating function. Then the proof proceeds in the same way as below using the polynomial dependence of the generating function on the coefficients of  $H$  (proposition 1.1).

For the elementary construction of hamiltonians with divergent normalization mentioned at the end of the introduction, we refer the reader to the end of section 30 of [Si-Mo], and to Siegel's article [Si1].

We consider the problem of convergence or divergence of Birkhoff normal form. The first assertion of the theorem follows from the second. Actually, consider the set  $F_n \subset \mathcal{H}_Q$  of hamiltonians having a converging Birkhoff normal form with radius of convergence  $> 1/n$ , and bounded by 1 in the ball of radius  $1/n$ . This set  $F_n$  is closed, and

$$F = \bigcup_{n \geq 1} F_n$$

is the set of all hamiltonians in  $\mathcal{H}_Q$  having a convergent Birkhoff normal form (so this set is an  $F_\sigma$ -set). Moreover, the open set  $\mathcal{H}_Q - F_n$  is dense. Otherwise let  $H_1$  be a hamiltonian in the interior of  $F_n$ . Considering the complex (resp. real) affine subspace  $V = \{(1-t)H_0 + tH_1; t \in \mathbf{C} \text{ (resp. } \mathbf{R})\} \subset \mathcal{H}_Q$  we have, according to the second assertion in theorem 1, that the set of hamiltonians with converging Birkhoff normal form must have capacity zero (resp. Lebesgue measure 0). But on the other hand it contains a neighborhood of 1. Contradiction.

The real analytic result follows from the  $\mathbf{C}$ -analytic one by the observation that the intersection of a set of  $\Gamma$ -capacity 0 in  $\mathbf{C}^n$  with  $\mathbf{R}^n \subset \mathbf{C}^n$  has Lebesgue measure 0 (see [Ro] Lemma 2.2.7 p. 90).

We consider a complex finite dimensional affine subspace  $V$  of  $\mathcal{H}$ . According to the definition of  $\Gamma$ -capacity we are reduced to the case of a one dimensional subspace  $V \approx \mathbf{C}$ . We can parametrize linearly the coefficients of hamiltonians  $H \in V$  with a complex parameter  $t \in \mathbf{C}$ , and we denote  $H_t$  the corresponding hamiltonian in  $V$ . Note that the coefficients of  $H_t$  are linear functions of  $t$ .

We assume that the Birkhoff normal form of hamiltonians  $H_t$  corresponding to a set of values  $t \in E \subset \mathbf{C}$  of positive capacity (non-polar) are converging. We want to prove that all the other hamiltonians in  $V$  have converging Birkhoff normal form.

We have

$$F = \bigcup_{n \geq 1} F_n$$

where  $F_n$  the set of parameters  $t \in \mathbf{C}$  such that the hamiltonian  $H_t$  has a Birkhoff normal form  $K_t$  with radius of convergence larger or equal to  $1/n$  and  $K_t$  is bounded by 1 in this ball. So if  $F$  is non-polar, we have for some  $n \geq 1$  that  $F_n$  is not polar (and this set is also closed). If we denote

$$K_t(\xi, \eta) = \sum_i K_i(t)(\xi, \eta)^i ,$$

then, according to proposition 1.1, the coefficients  $K_i(t)$  depend polynomially on  $t$  with degree  $\leq |i| - 2$  (for  $|i| \geq 3$ ). Now, there exists  $\rho_0 > 0$  such that for all  $t \in F_n$ ,

$$\varphi(t) = \limsup_{|i| \rightarrow +\infty} |K_i(t)| \rho_0^{-|i|} < +\infty .$$

The function  $\varphi$  is lower semicontinuous, and

$$F_n = \bigcup_m L_m$$

where  $L_m = \{z \in F_n; \varphi(t) \leq m\}$  is closed. By Baire theorem for some  $p$ ,  $L_m$  has non-empty interior (with respect to  $F_n$ ), thus this  $L_m$  has positive capacity. Finally we found a compact set  $C = L_m$  of positive capacity such that there exists  $\rho_1 > 0$  such that for any  $t \in C$  and all  $i \in \mathbf{N}^n$ ,

$$|K_i(t)| \leq \rho_1^{|i|} .$$

Using Bernstein's lemma and proposition 1.1 we get that for any compact set  $C_0 \subset \mathbf{C}$  we have for  $|i| \geq 3$ ,

$$\|K_i\|_{C^0(C_0)} \leq \rho^{|i|-2} \rho_1^{|i|} ,$$

for some constant  $\rho$  depending only on  $C_0$ . Thus  $K_t$  is converging for any  $t \in \mathbf{C}$ .

### c) Proof of theorem 2.

The proof of theorem 2 goes along the same lines than the proof of theorem 1, using the polynomial dependence of universal formal first integrals proved in corollary 1.4.

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